2023-2024 Term 2 MATH 116

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Appendix E Sigma Notation

Definition

A sequence is a set of objects **ordered** by positive integers. (These objects are usually numbers.) A sequence is said to be **finite** if it is finite as a set. A sequence is said to be **infinite** if it is not a finite sequence.

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Given a finite sequence $a_m, a_{m+1}, \cdots, a_n$ (where m and n are positive integers with $m \leq n$), we use the following **sigma notation** for their sum:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n.$$

Example

$$\sum_{i=5}^{7} 6 = 6 + 6 + 6, \quad \sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2.$$

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The Administration of Math 116

- **Course Coordinator**: **Mohamad Alwan**, Dept of Math and Statistics (Course and exam policy, exam delivery methods)
- **WebAssign and Lab Coordinator: Amos Lee**, Dept of Math and Statistics (Issues related to WebAssign and Canvas, Math 116 gradebook)
- Instructors: Teach math and answer math questions
- **Remarks:**
  - **Undergraduate Committee Chairs**: **Gary Au** and **Shahedul Khan**, Dept of Math and Statistics (Prerequisite requirements and Math career consultation)
  - **Department Chair: Artur Sowa** (final decision maker)

Rules of Summation

- Let c be a constant that is independent of the index i. Then $\sum_{i=m}^{n} c = c \cdot (\text{the number of terms}) = c(n-m+1)$.
- $\sum_{i=m}^{n}(a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$.
- $\sum_{i=m}^{n}(a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$.
- $\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$.

Theorem

$\sum_{i=1}^{n} i = n(n+1)/2.$

Theorem

$\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6.$

Theorem

$\sum_{i=1}^{n} i^3 = [n(n+1)/2]^2.$

Theorem

$\sum_{i=1}^{n} x^i = x(1-x^n)/(1-x) \text{ for } x \neq 1.$

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Example

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Find the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \left[\left(\frac{i}{n} \right)^2 + 1 \right].$$

Example

Find the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left[\left(\frac{2i}{n} \right)^3 + 5 \left(\frac{2i}{n} \right) \right].$$

5-1 The Area Problem

Example

Find the area under the curve $y = x^2$ from $x = 0$ to $x = 1$.

Definition

Let f be a **nonnegative**, **continuous** function on an interval $[a, b]$. Let

$$\Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x \qquad \text{for } i = 0, 1, 2, \cdots, n.$$

Choose a point x_i^* from the i-th closed subinterval $[x_{i-1}, x_i]$. Define the **Riemann sum** and the **area** under the curve $y = f(x)$, $a \leq x \leq b$, by

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \begin{cases} \textbf{upper sum} \text{ if one chooses} & f(x_i^*) = \max\limits_{x_{i-1} \leq x \leq x_i} f(x), \\ \textbf{lower sum} \text{ if one chooses} & f(x_i^*) = \min\limits_{x_{i-1} \leq x \leq x_i} f(x). \end{cases}$$

$$\textbf{The area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

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5-1 The Distance Problem

Definition

An object moves with **continuous** velocity $f(t)$, where $a \leq t \leq b$ and $f(t) \geq 0$. Let

$$\Delta t = \frac{b-a}{n}, \quad t_i = a + i\Delta t \qquad \text{for } i = 0, 1, 2, \cdots, n.$$

Choose a point t_i^* from the i-th closed subinterval $[t_{i-1}, t_i]$. Define the **distance** traveled during the time interval $[a, b]$ by

$$\textbf{The distance} = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i^*)\Delta t.$$

5-2 The Definite Integral

Definition

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Choose a point x_i^* from the i-th closed subinterval $[x_{i-1}, x_i]$. Define the **definite integral** (or simply **integration** or **integral**) of f over $[a, b]$ by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

- f not required to be positive; the sample point x_i^* can be arbitrary, for example, the mid-point $x_i^* = (x_{i-1} + x_i)/2$.
- Call $f(x)$ the **integrand**; a, b the **limits of integration**.
- Could use any letter in place of x without changing the value of the integral.

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Example

Express $\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^3 + x_i \sin x_i\right) \Delta x$ as an integral on the interval $[0, \pi]$.

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Example

Approximate the definite integral $\int_0^8 \sin\sqrt{x}\,dx$ using Riemann sums in the case of $n = 4$.

5-2 The Definite Integral

- c constant $\Rightarrow \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ and $\int_a^b c\,dx = c(b-a)$
- $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- If $a < c < b$ then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$
- When f takes both positive and negative values, then

$$\int_a^b f(x)\,dx = \text{the } \textbf{net area} \text{ of } f \text{ over } [a,b].$$

Example

Find the value of

$$\int_0^1 (\sqrt{1-x^2} - 6x)\,dx.$$

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Example

Find the value of

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5-2 The Definite Integral

- $f \geq 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq 0$
- $f \geq g$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
- $m \leq f \leq M$ on $[a, b] \Rightarrow m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

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5-3 The Fundamental Theorem of Calculus (FTC)

Theorem

Assume that f is continuous on $[a,b]$.

1. *The function*

$$g(x) = \int_a^x f(t)\,dt, \quad a \le x \le b,$$

is continuous on $[a,b]$ and differentiable on (a,b), and

$$g'(x) = f(x), \quad a < x < b.$$

2. *The value of the integral*

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is any ***anti-derivative*** *of f, that is, a function F such that $F' = f$.*

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Examples

Example

Find the derivative of

$$\int_0^x \sqrt{1+t^2}\,dt.$$

Example

Find the derivative of

$$\int_0^{x^4} \sec t\,dt.$$

Example

Evaluate

$$\int_3^6 \frac{1}{x}\,dx.$$

Announcement

Fact

The two midterm tests and the final exam this term for Math 116 are changed to ***online exams via WebAssign*** *that students can write from wherever they reside. Mr. Amos Lee will announce the details on Canvas.*

Examples

Example

What's wrong with the calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\, dx = -\frac{1}{x} \Big|_{x=-1}^{x=3} = -\frac{1}{3} - 1 < 0.$$

Example

Find the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \left[\left(\frac{i}{n} \right)^2 + 1 \right].$$

Example

Find the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left[\left(\frac{2i}{n} \right)^3 + 5 \left(\frac{2i}{n} \right) \right].$$

5-4 The Indefinite Integral

Definition

The notation

$$F(x) = \int f(x)\, dx \quad \text{means} \quad F'(x) = f(x).$$

Example

$$\int x^2\, dx = \frac{x^3}{3} + \text{constant}$$

$$\int \sin x\, dx = -\cos x + \text{constant}$$

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The notation

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5-4 The Indefinite Integral

Examples

(The Table of Indefinite Integrals Part I)

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + \text{constant} \quad (n \neq -1)$$

$$\int \frac{1}{x} \, dx = \ln|x| + \text{constant}$$

$$\int e^x \, dx = e^x + \text{constant}, \quad \int b^x \, dx = \frac{b^x}{\ln b} + \text{constant}$$

5-4 The Indefinite Integral

Examples

(The Table of Indefinite Integrals Part II)

$$\int \sin x \, dx = -\cos x + \text{constant}, \quad \int \cos x \, dx = \sin x + \text{constant}$$

$$\int \sec^2 x \, dx = \tan x + \text{constant}, \quad \int \csc^2 x \, dx = -\cot x + \text{constant}$$

$$\int \sec x \tan x \, dx = \sec x + \text{constant}, \quad \int \csc x \cot x \, dx = -\csc x + \text{constant}$$

$$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + \text{constant}, \quad \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + \text{constant}$$

5-4 The Indefinite Integral

Example

Define

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Then we have

$$\int \sinh x \, dx = \cosh x + \text{constant}, \quad \int \cosh x \, dx = \sinh x + \text{constant}.$$

Example

Compute

$$\int \frac{\cos\theta}{\sin^2\theta} \, d\theta.$$

Example

Evaluate

$$\int_0^2 2x^3 - 6x + \frac{3}{1+x^2} \, dx.$$

5-4 The Indefinite Integral

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Example

Evaluate

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt.$$

5-4 The Indefinite Integral: Applications

Theorem

(Net Change Theorem) If F is differentiable on some open interval that contains $[a, b]$, then

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

This is a reformulation of the FTC.

Example

An object moves alone the real line with **position** $s(t)$, then its **velocity** is $v(t) = s'(t)$, so

$$\textbf{displacement} \text{ during the time period } [t_1, t_2] = \int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1);$$

$$\textbf{distance} \text{ traveled during the time period } [t_1, t_2] = \int_{t_1}^{t_2} |v(t)|\,dt.$$

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Example

A particle moves on the real line with $v(t) = t^2 - t - 6$.

1. Find the displacement of the particle during the time period $1 \leq t \leq 4$.
2. Find the distance traveled during this time period.

5-5 The Substitution Rule

Example

$\int 2x\sqrt{1+x^2}\,dx$

Fact

(The Substitution Rule) If $u = g(x)$ is differentiable and its range is an interval I on which f is continuous, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example

$\int x^3 \cos(x^4+2)\,dx$

Example

$\int \sqrt{2x+1}\,dx$

5-5 The Substitution Rule

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Example

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5-5 The Substitution Rule

Example

$\int \frac{x}{\sqrt{1-4x^2}}\,dx$

Example

$\int e^{5x}\,dx$

Example

$\int x^5\sqrt{x^2+1}\,dx;\ u = x^2+1$

Example

$\int \tan x\,dx$

Example

$\int_1^2 \frac{1}{(3-5x)^2}\,dx$

5-5 The Substitution Rule

Example

$\int_1^e \frac{\ln x}{x}\, dx$

Fact

(Symmetry) Suppose f is continuous on $[-a, a]$.

1. *If $f(-x) = f(x)$ for all x, then $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$.*
2. *If $f(-x) = -f(x)$ for all x, then $\int_{-a}^{a} f(x)\, dx = 0$.*

Example

$\int_{-2}^{2} x^6 + 1\, dx$

Example

$\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\, dx = 0.$

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Example

$\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\,dx = 0.$

6-1 The Area Between Curves

Fact

The area bounded by the continuous curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$ is given by

$$area = \int_a^b |f(x) - g(x)|\, dx.$$

Example

Find the area of the region bounded by $y = e^x$, $y = x$, $x = 0$, $x = 1$.

Example

Find the area enclosed by parabolas $y = x^2$ and $y = 2x - x^2$.

Example

Find the area bounded by $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/2$.

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6-1 The Area Between Curves

Fact

Some regions are best treated by regarding x as a function in y. The area bounded by the continuous curves $x = f(y)$, $x = g(y)$, and the lines $y = c$, $y = d$ is given by

$$\text{area} = \int_c^d |f(y) - g(y)|\, dy.$$

Example

Find the area enclosed by $y = x - 1$ and $y^2 = 2x + 6$.

6-2 Volumes

Fact

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S through x and perpendicular to the x-axis is a continuous function $A(x)$, then

$$\text{volume of } S = \int_a^b A(x)\, dx.$$

Example

Find the volume of a ball with radius r.

$$A(x) = \pi y^2 = \pi(r^2 - x^2), \quad -r \leq x \leq r.$$

6-2 Volumes

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Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S through x and perpendicular to the x-axis is a continuous function $A(x)$, then

$$\textit{volume of } S = \int_a^b A(x)\,dx.$$

Example

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

$$A(x) = \pi(\sqrt{x})^2, \quad 0 \leq x \leq 1.$$

6-2 Volumes

Fact

Let S be a solid that lies between $y = c$ and $y = d$. If the cross-sectional area of S through y and perpendicular to the y-axis is a continuous function $A(y)$, then

$$\text{volume of } S = \int_c^d A(y)\, dy.$$

Example

Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about y-axis.

$$A(y) = \pi x^2 = \pi(\sqrt[3]{y})^2, \quad 0 \leq y \leq 8.$$

6-2 Volumes

Example

The region R enclosed by $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

$$A(x) = \pi x^2 - \pi (x^2)^2, \quad 0 \leq x \leq 1.$$

Example

Rotate the same region R about the horizontal line $y = 2$ and find the volume of the solid of revolution whose cross-section is a washer with the inner radius $2 - x$ and the outer radius $2 - x^2$.

6-2 Volumes

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Rotate the same region R about the horizontal line $y = 2$ and find the volume of the solid of revolution whose cross-section is a washer with the inner radius $2 - x$ and the outer radius $2 - x^2$.

Example

The same region R enclosed by $y = x$ and $y = x^2$ is now rotated about the vertical line $x = -1$. Find the volume of the solid of revolution whose cross-section is now a washer with the inner radius $1 + y$ and the outer radius $1 + \sqrt{y}$.

6-2 Volumes

Example

A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder, while the other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

Hint: Place the x-axis along the diameter where the planes meet and place the y-axis on the first plane, then the base of the solid is a semicircle $y = \sqrt{16 - x^2}$, $-4 \le x \le 4$. Then the cross-section perpendicular to the x-axis at x is a triangle whose base is $y = \sqrt{16 - x^2}$ and the height $y \tan 30^\circ$. Thus,

$$A(x) = \frac{16 - x^2}{2\sqrt{3}}, \quad -4 \le x \le 4.$$

6-3 Volumes by Cylindrical Shells

Fact

Let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$ where $f(x) \geq 0$, $y = 0$, $x = a$, $x = b$, where $b > a \geq 0$. Then

$$\text{volume of } S = \int_a^b 2\pi x f(x)\, dx.$$

Example

Find the volume of S where the region is bounded by $y = f(x) = 2x^2 - x^3$ and $y = 0$.

$$\text{volume of } S = \int_a^b \underbrace{2\pi x}_{\text{circumference}} \underbrace{f(x)}_{\text{height}} \underbrace{dx}_{\text{thickness}}.$$

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$$\text{volume of } S = \int_a^b 2\pi x f(x)\, dx.$$

Example

Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

$$\text{height } f(x) = x - x^2$$

6-3 Volumes by Cylindrical Shells

Fact

Let S be the solid obtained by rotating about ***the x-axis*** *the region bounded by $x = g(y)$ where $g(y) \geq 0$, $x = 0$, $y = c$, $y = d$, where $d > c \geq 0$. Then*

$$\text{volume of } S = \int_c^d 2\pi y g(y)\, dy.$$

Example

Find the volume of the solid obtained by rotating about **the x-axis** the region under the curve $y = \sqrt{x}$ from 0 to 1.

$$\text{radius} = y, \text{ circumference} = 2\pi y, \text{ height} = 1 - y^2$$

$$\text{volume} = \int_0^1 (2\pi y)(1 - y^2)\, dy$$

6-3 Volumes by Cylindrical Shells

Example

Find the volume of the solid obtained by rotating the region under the curve $y = x - x^2$ and $y = 0$ about **the line** $x = 2$.

$$\text{radius} = 2 - x,\ \text{circumference} = 2\pi(2 - x),\ \text{height} = x - x^2$$

$$\text{volume} = \int_0^1 2\pi(2 - x)(x - x^2)\,dx$$

6-4 Work

Example

An object moves along the x-axis in the positive direction. A each point x a force $f(x)$ acts continuously at the object. The work done in moving the object from $x = a$ to $x = b$ is

$$\text{work} = \int_a^b f(x)\, dx.$$

Unit of work: newton-meter (J) or foot-pound (ft-lb).

6-4 Work

Example

A force of 40 (newton) is needed to hold a spring that has been stretched from its natural length of 10 (cm) to a length of 15. How much work is done in stretching the spring from 15 to 18?

$$f(x) = kx \qquad \text{Hooke's Law}$$

The amount stretched is $15 - 10 = 5$ cm $= 0.05\,\text{m}$.
$f(0.05) = 40 \Rightarrow k = 800$.

$$W = \int_{0.05}^{0.08} 800x\,dx = 1.56\,J.$$

6-4 Work

Example

A 200 (lb) cable is 100 (ft) long and hangs vertically from the top of a building. Set the top of the building to be the origin and the x-axis pointing downward. Partition the cable into n small pieces of uniform length Δx, and let x_i^* denote a point in the ith such small piece. Assume the cable is made of uniform density so that it weighs 2 per foot (lb/ft), so the weight of the ith part is $2\Delta x$ (lb).

$$\text{work done on the } i\text{th part} = (\underbrace{2\Delta x}_{\text{force against gravity}}) \cdot \underbrace{x_i^*}_{\text{distance}}$$

Overall, the work is needed to lift the cable to the top of the building is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2x_i^* \Delta x = \int_0^{100} 2x\, dx.$$

6-4 Work

Example

A water tank has the shape of an inverted circular cone with height 10 (m) and base radius 4 (m). It is filled with water to a height of 8 (m). Find the work required to empty tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg$/m^3$.)

Hint: Measure depth from the top of the tank by placing $x = 0$ at there, and partition the (vertical) interval $[2, 10]$ into n subintervals and choose x_i^* from the i-th one, so that the water is divided into n layers.Then the i-th layer is approximated by a circular cylinder with radius r_i and height $\Delta x = 8/n$, where

$$\frac{r_i}{10 - x_i^*} = \frac{4}{10}. \Rightarrow volume = \frac{4\pi}{25}(10 - x_i^*)\Delta x.$$

6-4 Work

Example

(Water tank problem continued) The density of water is 1000 (kg/m^3), the gravitational constant $g = 9.8$, and so mass m_i = density $\times$volume $=160\pi(10-x_i^*)\Delta x$. The force F_i is $9.8m_i$, and the work done for the i-th part is $W_i = F_i x_i^*$.

$$W = \lim_{n\to\infty} W_i = \int_2^{10} 1568\pi x(10-x)^2\, dx.$$

6-5 Average Value of a Function

Definition

The average value of a function f on $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Theorem

(Mean Value Theorem) If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example

Find such a c for $f(x) = 1 + x^2$ on $[-1, 2]$.

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7-1 Integration by Parts

Definition

(Integration by Parts)

$$\int u\,dv = uv - \int v\,du.$$

Example

Find $\int x \sin x\,dx$.

Example

Find $\int \ln x\,dx$.

Example

Find $\int t^2 e^t\,dt$.

Example

Find $\int e^x \sin x\,dx$.

7-1 Integration by Parts

Definition

(Integration by Parts)

$$\int u\,dv = uv - \int v\,du.$$

Example

Find $\int_0^1 \tan^{-1} x\,dx$. $u = \tan^{-1} x$, $dv = dx$

Example

$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$

Example

$\int \cos^n x\,dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$

7-2 Trig Integrals

Example

Find $\int \cos^3 x\, dx$. $\cos^2 x + \sin^2 x = 1$

Example

Find $\int \sin^5 x \cos^2 x\, dx$.

Example

Find $\int_0^\pi \sin^2 x\, dx$ $\sin^2 x = (1 - \cos 2x)/2$

Example

Find $\int \sin^4 x\, dx$.Use the reduction formula.

7-2 Trig Integrals

Fact

To evaluate $\int \sin^m x \cos^n x \, dx$*:*

1. *If n is odd, separate one* $\cos x$ *out and use* $\cos^2 x = 1 - \sin^2 x$.
2. *If m is odd, separate one* $\sin x$ *out and use* $\cos^2 x = 1 - \sin^2 x$.
3. *If both* m, n *are even, use*

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \cos^2 x = \frac{1 + \cos 2x}{2}, \sin x \cos x = \frac{\sin 2x}{2}.$$

Example

$\int \sin^4 x \cos^4 x \, dx$

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Example

$\int \sin^4 x \cos^4 x\, dx$

7-2 Trig Integrals

Example

Find $\int \tan^6 x \sec^4 x \, dx$. $\sec^2 x = 1 + \tan^2 x$, $u = \tan x$, $du = \sec^2 x \, dx$

Example

Find $\int \tan^5 x \sec^7 x \, dx$. $u = \sec x$, $du = \sec x \tan x \, dx$

Example

Find $\int \tan x \, dx = \ln|\sec x| + C$, $\tan x = \frac{\sin x}{\cos x}$

Example

Find $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

7-2 Trig Integrals

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To evaluate $\int \tan^m x \sec^n x \, dx$:

1. *If n is even, separate one* $\sec^2 x$ *out and use* $\sec^2 x = 1 + \tan^2 x$.
2. *If m is odd, separate one* $\sec x \tan x$ *out and use* $\tan^2 x = \sec^2 x - 1$.

Example

$\int \tan^3 x \, dx$; use $\tan^2 x = \sec^2 x - 1$ first then follow (1).

Example

$\int \sec^3 x \, dx$; use integration by parts: $u = \sec x$, $dv = \sec^2 x \, dx$

$$-\int \tan^2 x \sec x \, dx = -\int (\sec^2 x - 1) \sec x \, dx = \text{"} - \text{"} \int \sec^3 x \, dx + \int \sec x \, dx$$

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7-2 Trig Integrals

Fact

Product-Sum Formulas

1. $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
2. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
3. $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Example

$\int \sin 4x \cos 5x \, dx$

7-2 Trig Integrals

Fact

Product-Sum Formulas

1. $\sin A \cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$
2. $\sin A \sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$
3. $\cos A \cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

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7-3 Trig Substitution

Fact

Table of Trig Substitutions

1. $\sqrt{a^2 - x^2} \Rightarrow x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $1 - \sin^2\theta = \cos^2\theta$.
2. $\sqrt{a^2 + x^2} \Rightarrow x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $1 + \tan^2\theta = \sec^2\theta$.
3. $\sqrt{x^2 - a^2} \Rightarrow x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ *or* $\pi \le \theta < \frac{3\pi}{2}$, $\sec^2\theta - 1 = \tan^2\theta$.

Example

$\int \frac{\sqrt{9-x^2}}{x^2}\,dx$, $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$, $\int \frac{x}{\sqrt{x^2+4}}\,dx$, $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$.

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1. $\sqrt{a^2 - x^2} \Rightarrow x = a\sin\theta,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ 1 - \sin^2\theta = \cos^2\theta.$
2. $\sqrt{a^2 + x^2} \Rightarrow x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ 1 + \tan^2\theta = \sec^2\theta.$
3. $\sqrt{x^2 - a^2} \Rightarrow x = a\sec\theta,\ 0 \leq \theta < \frac{\pi}{2}$ *or* $\pi \leq \theta < \frac{3\pi}{2},\ \sec^2\theta - 1 = \tan^2\theta.$

Example

$\int \frac{\sqrt{9-x^2}}{x^2}\,dx,\ \int \frac{1}{x^2\sqrt{x^2+4}}\,dx,\ \int \frac{x}{\sqrt{x^2+4}}\,dx,\ \int \frac{1}{\sqrt{x^2-a^2}}\,dx.$

7-3 Trig Substitution

Fact

Table of Trig Substitutions

1. $\sqrt{a^2 - x^2} \Rightarrow x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 1 - \sin^2\theta = \cos^2\theta.$
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Example

$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx,\ \int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

7-4 Partial Fractions (Long Division Reduction)

Fact

Long Division Algorithm

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

Example

$$\int \frac{x^3 + x}{x - 1}\, dx$$

7-4 Partial Fractions (Distinct Linear Factors)

Fact

If the denominator $g(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$ *where no factor is repeated, then*

$$\frac{f(x)}{g(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

Example

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx$$

$$\int \frac{1}{x^2 - a^2}\, dx$$

7-4 Partial Fractions (Repeated Linear Factors)

Fact

If some factors are repeated, say, $(a_1x + b_1)^r$, *then one replaces*

$$\frac{A_1}{a_1x + b_1}$$

by

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r},$$

and do this for each repeated factor.

Example

$$\frac{x^3 - x + 1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$$

7-4 Partial Fractions (Distinct Irreducible Quadratic Factors)

Fact

If there is a irreducible quadratic factor $ax^2 + bx + c$, *then in addition to the partial fractions in previous cases, one adds*

$$\frac{Ax + B}{ax^2 + bx + c}.$$

Example

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

Note that $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$

7-4 Partial Fractions (Repeated Irreducible Quadratic Factors)

Fact

If there is a repeated irreducible quadratic factor $(ax^2+bx+c)^r$*, then in addition to the partial fractions in previous cases, one replaces*

$$\frac{Ax+B}{ax^2+bx+c}$$

by

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}.$$

Example

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

$$\int\frac{\sqrt{x+4}}{x}\,dx;\ u=\sqrt{x+4}$$

7-5 Integration Strategy

Example

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx;\ \frac{\tan^3 x}{\cos^3 x} = \underbrace{\tan^3 x \sec^3 x}_{u=\sec x} = \underbrace{\frac{\sin^3 x}{\cos^6 x}}_{u=\cos x}$$

Example

$$\int e^{\sqrt{x}}\,dx = 2\int ue^u\,du$$

Example

$$\int \frac{1}{x\sqrt{\ln x}}\,dx = \int \frac{1}{\sqrt{u}}\,du;\quad \int \underbrace{\sqrt{\frac{1-x}{1+x}}}_{=u}\,dx,\ \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

7-5 Integration Strategy

Example

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx;\ \frac{\tan^3 x}{\cos^3 x} = \underbrace{\tan^3 x \sec^3 x}_{u=\sec x} = \frac{\sin^3 x}{\underbrace{\cos^6 x}_{u=\cos x}}$$

Example

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Example

$$\int \frac{1}{x\sqrt{\ln x}}\,dx = \int \frac{1}{\sqrt{u}}\,du; \quad \int \underbrace{\sqrt{\frac{1-x}{1+x}}}_{=u}\,dx, \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

7-5 Integration Strategy

Example

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$$\int \frac{1}{x\sqrt{\ln x}}\,dx = \int \frac{1}{\sqrt{u}}\,du; \quad \int \underbrace{\sqrt{\frac{1-x}{1+x}}}_{=u}\,dx, \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

7-7 Approximation Integration

Fact

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x \qquad \textit{(left point approximation)}$$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\Delta x \qquad \textit{(right point approximation)}$$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x \qquad \textit{(midpoint approximation)}$$

7-7 Approximation Integration

Fact

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x \qquad \textit{(left point approximation)}$$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\Delta x \qquad \textit{(right point approximation)}$$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \left[\frac{f(x_{i-1})+f(x_i)}{2}\right]\Delta x \qquad \textit{(Trapezoidal Rule)}$$

$$Trapezoidal = average\ of\ the\ left\ and\ the\ right$$

7-7 Approximation Integration

Definition

The **error** in an approximation is defined to the

$$\text{the error} = \text{the exact value} - \text{the approximation.}$$

Example

The Trapezoidal Rule for $\int_1^2 \frac{1}{x}\,dx$ where $n = 5$ gives the approximation

$$T = 0.695635,$$

then

$$\text{error} = \int_1^2 \frac{1}{x}\,dx - T = \ln 2 - T = -0.002488.$$

7-7 Approximation Integration

Theorem

(Error Bounds) Suppose

$$|f''(x)| \leq K \quad \text{for } a \leq x \leq b.$$

Then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}, \quad |E_M| \leq \frac{K(b-a)^3}{24n^2},$$

where E_T and E_M denote respectively the errors in the Trapezoidal and Midpoint Rules.

Example

The Trapezoidal Rule of $\int_1^2 \frac{1}{x}\,dx$ for $n = 5$ yields

$$|E_T| \leq \frac{2(2-1)^3}{12 \cdot 5^2}.$$

7-7 Approximation Integration

Theorem

(Error Bounds) Suppose

$$|f''(x)| \leq K \quad \text{for } a \leq x \leq b.$$

Then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}, \quad |E_M| \leq \frac{K(b-a)^3}{24n^2},$$

where E_T and E_M denote respectively the errors in the Trapezoidal and Midpoint Rules.

Example

How large should we take n in order to guarantee that the Trapezoidal approximation for $\int_1^2 \frac{1}{x}\,dx$ is accurate within 0.0001?

$$|E_T| \leq \frac{2(2-1)^3}{12 \cdot n^2} < 0.0001$$

7-7 Approximation Integration

Theorem

*(Simpson's Rule) Assume n is an **even** number.*

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$$

*where n is **even** and $\Delta x = (b-a)/n$.*

$$pattern = 1,4,2,4,2,\cdots,4,2,4,1$$

Theorem

(Error Bound) Suppose

$$|f^{(4)}(x)| \le K, \qquad a \le x \le b.$$

Then

$$|E_S| \le \frac{K(b-a)^5}{180n^4}.$$

4-4 L'Hospital's Rule

Theorem

Let f and g be differentiable on an open interval I containing a. Suppose that

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x),$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \quad \textit{and} \quad \lim_{x\to a} g(x) = \pm\infty.$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right side exists (or is ∞ or $-\infty$).

Examples

$\lim_{x\to 1} \frac{\ln x}{x-1}$, $\lim_{x\to\infty} \frac{e^x}{x^2}$, $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$, $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

4-4 L'Hospital's Rule

Theorem

Let f and g be differentiable on an open interval I containing a. Suppose that

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or that

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Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right side exists (or is ∞ or $-\infty$).

Examples

$\lim_{x \to 0^+} x \ln x$, $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$, $\lim_{x \to \infty} (e^x - x)$

4-4 L'Hospital's Rule

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Let f and g be differentiable on an open interval I containing a. Suppose that

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or that

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Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right side exists (or is ∞ or $-\infty$).

Examples

$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$, $\lim_{x \to 0^+} x^x$

7-8 Improper Integrals

Definitions

Improper integrals of type 1 (over unbounded intervals):

1. $$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$
2. $$\int_{-\infty}^b f(x)\,dx = \lim_{t\to\infty} \int_t^b f(x)\,dx$$
3. $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

Example

Determine whether the (improper) integral $\int_1^\infty \frac{1}{x}\,dx$ is convergent or divergent. More generally, how about $\int_1^\infty \frac{1}{x^p}\,dx$ for $p > 0$?

7-8 Improper Integrals

Definitions

Improper integrals of type 2 (with discontinuities):

1. f is discontinuous at b:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

2. f is discontinuous at a:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

3. f is discontinuous at c, where $a < c < b$:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

7-8 Improper Integrals

Theorem

(Comparison Test) If $0 \leq g \leq f$, *then*

1. *If* $\int_a^\infty f(x)\,dx$ *is convergent, then* $\int_a^\infty g(x)\,dx$ *is convergent.*
2. *If* $\int_a^\infty g(x)\,dx$ *is divergent, then* $\int_a^\infty f(x)\,dx$ *is divergent.*

Example

$\int_0^\infty e^{-x^2}\,dx$ is convergent and $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent.

Example

$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$, $\int_0^{\pi/2} \sec x\,dx$, $\int_0^3 \frac{1}{x-1}\,dx$, $\int_0^1 \ln x\,dx$

8-1 Arc Length

Definition

If f' is continuous on $[a, b]$ then the length of the curve $y = f(x)$, $a \leq x \leq b$, is given by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Using Leibniz notation, the formula can be rewritten as

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx.$$

Example

Find L for $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.

8-1 Arc Length

Definition

If $g'(y)$ is continuous on $[c, d]$ then the length of the curve $x = g(y)$, $c \leq y \leq d$, is given by

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Using Leibniz notation, the formula can be rewritten as

$$L = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy.$$

Example

Find L for $y^2 = x$ between the points $(0, 0)$ and $(1, 1)$.

8-1 Arc Length

Definition

Given a curve $y = f(x)$, $a \le x \le b$, let

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt, \quad a \le x \le b,$$

be the arc length from point $(a, f(a))$ to $(x, f(x))$. $s(x)$ is called the **arc length function**. Note that FTC implies that

$$s'(x) = \sqrt{1 + [f'(x)]^2}, \quad \text{or equivalently,}$$

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx.$$

Example

Find $s(x)$ for $y = x^2 - \frac{1}{8}\ln x$ starting at the point $(1, 1)$.

8-2 Area of a Surface of Revolution

Definition

Consider the surface obtained by rotating the curve $y = f(x) \geq 0$, $a \leq x \leq b$, about the x-axis, the surface area is given by

$$S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx = \int_a^b 2\pi y\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx = \int_a^b 2\pi y\,ds.$$

For rotation about y-axis of $x = g(y)$, $c \leq y \leq d$, we have

$$S = \int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}\,dy = \int_c^d 2\pi x\sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy = \int_c^d 2\pi x\,ds.$$

Example

Find S when rotating $y = \sqrt{4-x^2}$, $-1 \leq x \leq 1$, about x-axis.

8-2 Area of a Surface of Revolution

Definition

Consider the surface obtained by rotating the curve $y = f(x) \geq 0$, $a \leq x \leq b$, about the x-axis, the surface area is given by

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Example

Find S when rotating $y = e^x$, $0 \leq x \leq 1$, about x-axis.

3-8 Exponential Growth and Decay

Definition

If $y(t)$ is the value of a quantity y at the time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time t, then

$$\frac{dy}{dt} = ky \quad \text{for some constant } k,$$

and the only solution for this differential equation is

$$y(t) = y(0)e^{kt}.$$

The constant k is called the **relative growth rate** of the quantity y.

Example

Suppose the growth rate of a certain population is proportional to the population size $P(t)$, and say, $P(0) = 2560$ and $P(10) = 3040$. Then the relative growth rate is $k = 0.017$ and $P(t) = 2560e^{kt}$.

3-8 Exponential Growth and Decay

Example

The half-life of a certain radioactive element is 1590 years.

1. Find a formula for the mass $m(t)$ of the element that remains after t years. Suppose $m(0) = 100$.
2. Find the mass $m(1000)$ after 1000 years.
3. When will the mass be reduced to 30?

Example

Newton's law of cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s),$$

where k is a constant and T_s is the (constant) temperature of surroundings. Make a change of variable $y(t) = T(t) - T_s$ to rewrite it as $y' = ky$.

Example

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where k is a constant and T_s is the (constant) temperature of surroundings. Make a change of variable $y(t) = T(t) - T_s$ to rewrite it as $y' = ky$.

3-8 Exponential Growth and Decay

Example

Denote by $A(t)$ the amount of a financial investment at time t. The continuous compounding of A with interest rate r is governing by the differential equation:

$$\frac{dA}{dt} = rA(t).$$

For example, \$1000 invested for 3 years at 6% interest rate will has its value

$$A(3) = 1000e^{(0.06)3} = 1197.22.$$

9-1 Modeling with Differential Equations

Example

The equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

shows that

1. If P is small, then

   $$\frac{dP}{dt} \approx kP. \text{ (Initially, the growth rate is proportional to } P.\text{)}$$

2. If $P > M$, then

   $$\frac{dP}{dt} < 0. \text{ (} P \text{ decreases if it ever exceeds the constant } M.\text{)}$$

9-1 Modeling with Differential Equations

Example

Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}, \quad c \text{ is any constant,}$$

satisfies the differential equation

$$y' = \frac{1}{2}\left(y^2 - 1\right).$$

Moreover, the solution of the equation $y' = \frac{1}{2}\left(y^2 - 1\right)$ satisfying the initial condition $y(0) = 2$ is

$$y = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t}.$$

9-3 Separable Equations

Definition

$$\frac{dy}{dx} = g(x)f(y)$$

Example

$$y' = \frac{x^2}{y^2}, \quad y(0) = 2.$$

Example

$$y' = \frac{6x^2}{2y + \cos y}$$

Example

$$y' = x^2 y$$

9-3 Separable Equations

Example

A water tank contains 20 kg of salt dissolved in 5000 L of water. Salted water that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/minute. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after 30 minutes?

$$y(t) = \text{amount of salt at time } t$$

$$y' = (\text{rate in}) \text{ - } (\text{rate out}), \quad y(0) = 20.$$

$$\text{rate in } = 0.03\frac{kg}{L} * 25\frac{L}{min} = 0.75\frac{kg}{min}$$

$$\text{rate out } = \frac{y(t)}{5000}\frac{kg}{L} * 25\frac{L}{min} = \frac{y(t)}{200}\frac{kg}{min}$$

To be continued

Fact

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Examples

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Example

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To be continued

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